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SINGULARITIES AND CLASSICAL LIMIT IN QUANTUM COSMOLOGY WITH SCALAR FIELDS

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Abstract

Minisuperspace models derived from Kaluza-Klein theories and low energy string theory are studied. They are equivalent to one and two minimally coupled scalar fields. The general classical and quantum solutions are obtained. Gaussian superposition of WKB solutions are constructed. Contrarily to what is usually expected, these states are sharply peaked around the classical trajectories only for small values of the scale factor. This behaviour is confirmed in the framework of the causal interpretation: the Bohmian trajectories of many quantum states are classical for small values of the scale factor but present quantum behaviour when the scale factor becomes large. A consequence of this fact is that these states present an initial singularity. However, there are some particular superpositions of these wave functions which have Bohmian trajectories without singularities. There are also singular Bohmian trajectories with a short period of inflation which grow forever. We could not find any non-singular trajectory which grows to the size of our universe.

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1 Introduction

One of the main motivations to study quantum cosmology is to investigate if quantum gravitational effects can avoid the singularities which are present in classical cosmological models [1]. If this is indeed the case for the initial singularity, the next step should be to find in what conditions the universe recovers its classical behaviour, yielding the large

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classical expanding universe we live in. In this paper we investigate these problems in the framework of minisuperspace models with scalar fields as sources of the gravitational field.

As a first example, we took a non-massive, minimally coupled scalar field, in a Friedman-Robertson-Walker universe with spacelike sections with positive constant curvature. This model can be viewed as an effective multidimensional theory where the scalar field is understood as the scale factor of internal dimensions [2], or as a Brans-Dicke model redefined by a conformal transformation [3]. We were able to find the general classical solutions. All of them present initial and final singularities. The model is quantized in the Dirac way, with arbitrary factor ordering, and the general solution of the corresponding Wheeler-DeWitt equation is found. To interpret the solutions, we first adopted the 'peak interpretation', where a prediction is made when the wave function is sharply peaked in a region and almost zero outside this region [4]. A gaussian superposition of WKB solutions was constructed. By employing the stationary phase condition, we were able to show that this superposition is sharply peaked around the classical trajectory only for small values of the scale factor. Hence, contrarily to what is usually expected, the classical limit is recovered for small values of the scale factor, not for large ones. A consequence of this fact is that the initial classical singularities continue to be present at the quantum level. In order to confirm this strange behaviour, we also adopted an alternative interpretation of quantum mechanics which was not constructed for cosmology but which can be easily applied to a single system: it is the causal or the Bohm-de Broglie interpretation of quantum mechanics [5]. It is completely different from the others because it is an ontological interpretation of quantum mechanics. In the case of non-relativistic particles, the quantum particles follows a real trajectory, independently of any observations, and it is accompanied by a wave function. The quantum effects are brought about by a quantum potential, which can be derived from the Schrödinger equation. It is a rather simple interpretation which can be easily applied to minisuperspace models [6]. In this case, the Schrödinger equation is replaced by the Wheeler-DeWitt equation, and the quantum trajectories are the time evolutions of the metric and field variables, which obey a Hamilton-Jacobi equation with an extra quantum potential term. The application of this interpretation to some of the quantum solutions of our problem shows exactly the same behaviour as found previously: the Bohmian trajectories behave classically for small values of the scale factor while the quantum behaviour appears when the scale factor becomes large. Singularities are still present. However, when we make superpositions of these wave functions, the initial singularity disappears for some special cases, but none of these special trajectories grows to the size of our universe.

The other case studied involves two minimally coupled scalar fields. They can be viewed as a tree level effective action of string theory where the second scalar field comes from the Kalb-Rammond field [7]. They can also be understood as generalized Brans-Dicke type models, which can be derived from compactification of multidimensional theories with external gauge fields [8]. The results obtained in this case were analogous to the preceding one. Along the lines of the peak interpretation, gaussian WKB superpositions predicts a classical universe for small values of the scale factor because they are peaked around the classical trajectories in this region. Adopting the causal interpretation to in-

vestigate the singularity problem, we found, as before, that many of the solutions present classical behaviour when the scale factor is small (and hence singularities) but behaves quantum mechanically when the scale factor becomes large.

This paper is organized as follows: in the next section we describe the classical minisuperspace models of both one and two scalar fields models, presenting their general classical solutions. In section 3 we quantize these models obtaining their corresponding Wheeler-DeWitt equations and their respective general solutions. In section 4, the gaussian superpositions of WKB solutions are constructed and their peak along the classical trajectories are exhibited. In section 5, the causal interpretation of quantum cosmology is shortly reviewed and applied to the quantum solutions. We end with comments and conclusions.

2 The Classical Models

Models with two scalar fields that interact non-trivially between themselves can be obtained from different theoretical contexts. Considering Kaluza-Klein supergravity theories, keeping just the bosonic sector, and reducing to four dimensions, leads to effective actions with gravity plus two scalar fields, one of them coupled non-minimally to the Einstein-Hilbert lagrangian; the two scalar fields have an interaction between them. More generally, every time we consider multidimensional models with gauge fields, and reduce them to four dimensions, we find such structure. String theories, in particular, have an effective action in four dimensions given by the expression,

$$L = \sqrt{-g}e^{-\phi}\Big(R + \phi_{;\rho}\phi^{;\rho} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\Big) \quad , \tag{1}$$

where ϕ is a dilaton field and $H_{\mu\nu\lambda}$ is a Kalb-Ramond field which in four dimensions is equivalent to a scalar field ξ

In order to keep contact with this variety of models, all of them having great importance in high energy conditions, we will consider the general lagrangian

$$L = \sqrt{-g}\Big(\phi R - \omega\frac{\phi_{;\rho}\phi^{;\rho}}{\phi} - \frac{\xi_{;\rho}\xi^{;\rho}}{\phi}\Big) \quad , \tag{2}$$

where ω is a coupling constant. We remark the non-trivial interaction between ϕ and ξ. For the string effective action, $\omega = -1$ and for Kaluza-Klein theories $\omega = \frac{1-d}{d}$, where d is the dimension of internal compact spacelike dimensions. If we perform a conformal transformation such that $g_{\mu\nu} = \phi^{-1}\bar{g}_{\mu\nu}$, we obtain the lagrangian

$$L = \sqrt{-g}\Big[R - (\omega + \frac{3}{2})\frac{\phi_{;\rho}\phi^{;\rho}}{\phi^2} - \frac{\xi_{;\rho}\xi^{;\rho}}{\phi^2}\Big] \quad , \tag{3}$$

where the bars have been suppressed. From Eq. (3) we deduce the field equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\kappa}{\phi^2}\Big(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\rho}\phi^{;\rho}\Big) + \frac{1}{\phi^2}\Big(\xi_{;\mu}\xi_{;\nu} - \frac{1}{2}g_{\mu\nu}\xi_{;\rho}\xi^{;\rho}\Big) \quad , \tag{4}$$

$$\Box\phi - \frac{\phi^{;\rho}\phi_{;\rho}}{\phi} + \frac{\xi_{;\rho}\xi^{;\rho}}{\kappa\phi} = 0 \quad , \tag{5}$$

$$\Box\xi - 2\frac{\xi_{;\rho}\phi^{;\rho}}{\phi} = 0 \quad , \tag{6}$$

where $\kappa = \omega + \frac{3}{2}$.

We consider now the Robertson-Walker metric

$$ds^2 = -N^2 \mathrm{d}t^2 + \frac{a(t)^2}{1 + \frac{\epsilon}{4}r^2}[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2)], \tag{7}$$

where the spatial curvature ϵ takes the values 0, 1,-1. The equations of motion are, for $N = 1$,

$$3(\frac{\dot{a}}{a})^2 + \frac{3\epsilon}{a^2} = \frac{\kappa}{2}(\frac{\dot{\phi}}{\phi})^2 + \frac{1}{2}(\frac{\dot{\xi}}{\phi})^2 \quad , \tag{8}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{\dot{\phi}^2}{\phi} + \frac{\dot{\xi}^2}{\kappa\phi} = 0 \quad , \tag{9}$$

$$\ddot{\xi} + 3\frac{\dot{a}}{a}\dot{\xi} - 2\frac{\dot{\phi}}{\phi}\dot{\xi} = 0 \quad . \tag{10}$$

We will be interested in the case $\epsilon = 1$. In what follows we will consider separately the cases where $\xi =$ const. and $\xi \neq$ const..

2.1 One scalar field minimaly coupled to gravity

Henceforth, we consider in Eqs. (8,9,10) $\xi =$ constant. The solutions of the resulting equations can be easily found if we reparametrize the time coordinate as $dt = a^3 d\theta$. The integration procedure is standard, and we just give the final results:

$$\phi = Ae^{\theta + B} \quad ; \tag{11}$$

$$a = \sqrt{A}\Big(\frac{\kappa}{6}\Big)^{\frac{1}{4}} \frac{1}{\sqrt{\cosh\Big(\sqrt{\frac{2}{3}}A\sqrt{\kappa}(\theta + C)\Big)}} \quad . \tag{12}$$

In these expressions, A, B and C are integration constants. The universe expands from an initial singularity untill a maximum size and then contract to a final singularity. Note that $a \propto t^{\frac{1}{3}}$ for small a. For $A = 1$ and $B = C$, we obtain the implicit relation:

$$a(\phi) = \Big[\frac{2}{3}\kappa\Big]^{\frac{1}{4}} \sqrt{\frac{\phi^{\sqrt{\frac{2}{3}\kappa}}}{1 + \phi^{2\sqrt{\frac{2}{3}\kappa}}}} \quad . \tag{13}$$

2.2 Two scalar fields coupled to gravity

Considering the fields ϕ and ξ in equations (8, 9,10), and using again the same parameter θ as defined previously, we find the following solutions:

$$\xi = A + \frac{C}{B}\kappa \tanh\Big(\frac{C(\theta + D)}{\kappa}\Big) \quad ; \tag{14}$$

$$\phi = \frac{C}{B}\frac{1}{\cosh\Big(\frac{C(\theta+D)}{\sqrt{\kappa}}\Big)} \quad ; \tag{15}$$

$$a = \frac{\sqrt{C}}{6^{\frac{1}{4}}}\frac{1}{\sqrt{\cosh\Big(\sqrt{\frac{2}{3}}|C|(\theta + E)\Big)}} \quad . \tag{16}$$

In these expressions A, B, C, D and E are constants. The qualitative behaviour of the scale factor is the same as in the preceding case (compare (12) with (16)). Again we have $a \propto t^{\frac{1}{3}}$ when a is small. For $A = 0$, $B = C$, $D = E$ and $\kappa = \frac{3}{2}$, we can find a simple implicit relation between a, ϕ and ξ:

$$\phi(a) = \frac{1}{|C|}\sqrt{6}a^2 \quad ; \tag{17}$$

$$a(\xi) = \sqrt{|C|}(\frac{1}{6} - \frac{\xi^2}{9})^{\frac{1}{4}} \quad ; \tag{18}$$

$$\phi(\xi) = \sqrt{1 - \frac{2}{3}\xi^2} \quad . \tag{19}$$

These implicit classical relations, together with Eq. (13), will be compared with the trajectory on which the semi-classical wave function of the corresponding quantum model is peaked.

3 Quantum Solutions in Minisuperspace

We return to the lagrangian (3) and we insert on it the metric (7). The action takes the form

$$S = \int L dt \tag{20}$$

where

$$L = \frac{12a\dot{a}^2}{N} - (3 + 2\omega)\frac{a^3\dot{\phi}^2}{N\phi^2} - 2\frac{a^3\dot{\xi}^2}{N\phi^2} - 12Na \quad . \tag{21}$$

From (21) we obtain the conjugate momenta,

$$\pi_a = 24\frac{a\dot{a}}{N} \quad , \tag{22}$$

$$\pi_\phi = -2(3 + 2\omega)\frac{a^3\dot{\phi}}{N\phi^2} \quad , \tag{23}$$

$$\pi_\xi = -4\frac{a^3\dot{\xi}}{N\phi^2} \quad . \tag{24}$$

We can now construct the hamiltonian H, which takes the form

$$H = N\left[\frac{\pi_a^2}{48a} - \frac{\phi^2\pi_\phi^2}{4(3+2\omega)a^3} - \frac{\phi^2\pi_\xi^2}{8a^3} + 12a\right] \equiv N\mathcal{H} \quad . \tag{25}$$

Variation of N yields the first class constraint $\mathcal{H} \approx 0$. The Dirac quantization procedure yields the Wheeler-DeWitt equation by imposing the condition:

$$\hat{\mathcal{H}}\Psi = 0 \tag{26}$$

and performing the substitutions

$$\pi_a^2 \rightarrow -\frac{\partial^2}{\partial a^2} - \frac{p}{a}\frac{\partial}{\partial a} \quad , \tag{27}$$

$$\pi_\phi^2 \rightarrow -\frac{\partial^2}{\partial \phi^2} - \frac{q}{\phi}\frac{\partial}{\partial \phi} \quad , \tag{28}$$

$$\pi_\xi^2 \rightarrow -\frac{\partial^2}{\partial \xi^2} \quad , \tag{29}$$

where p and q are ordering factors. We have set $\hbar = 1$. The Wheeler-DeWitt equation in the minisuperspace reads

$$\frac{a^2}{12}\left[\Psi_{aa} + \frac{p}{a}\Psi_a\right] - \frac{\phi^2}{(3+2\omega)}\left[\Psi_{\phi\phi} + \frac{q}{\phi}\Psi_\phi\right] - \frac{\phi^2}{2}\Psi_{\xi\xi} = V_\Psi(a)\Psi \quad , \tag{30}$$

where $V_\Psi(a) = 48a^4$.

We will solve this equation for the cases $\xi = 0$ (one scalar field) and $\xi \neq 0$ (two scalar fields).

3.1 Solutions with one scalar field

Discarding the field ξ, we have to solve the equation,

$$\frac{a^2}{12}\left[\Psi_{aa} + \frac{p}{a}\Psi_a\right] - \frac{\phi^2}{(3+2\omega)}\left[\Psi_{\phi\phi} + \frac{q}{\phi}\Psi_\phi\right] = V_\Psi(a)\Psi \quad . \tag{31}$$

Supposing the separability of this equation, we can write $\Psi(a,\phi) = \alpha(a)\beta(\phi)$ leading to two ordinary differential equations for α and β:

$$\alpha_{aa} + \frac{p}{a} = V_\alpha(a)\alpha \quad , \tag{32}$$

$$\beta_{\phi\phi} + \frac{q}{\phi}\beta_\phi = V_\beta(\phi)\beta \quad , \tag{33}$$

where

$$V_\alpha = 12\left(48a^2 - \frac{k}{a^2}\right) \quad , \quad V_\beta(\phi) = -(3+2\omega)\frac{k}{\phi^2} \quad , \tag{34}$$

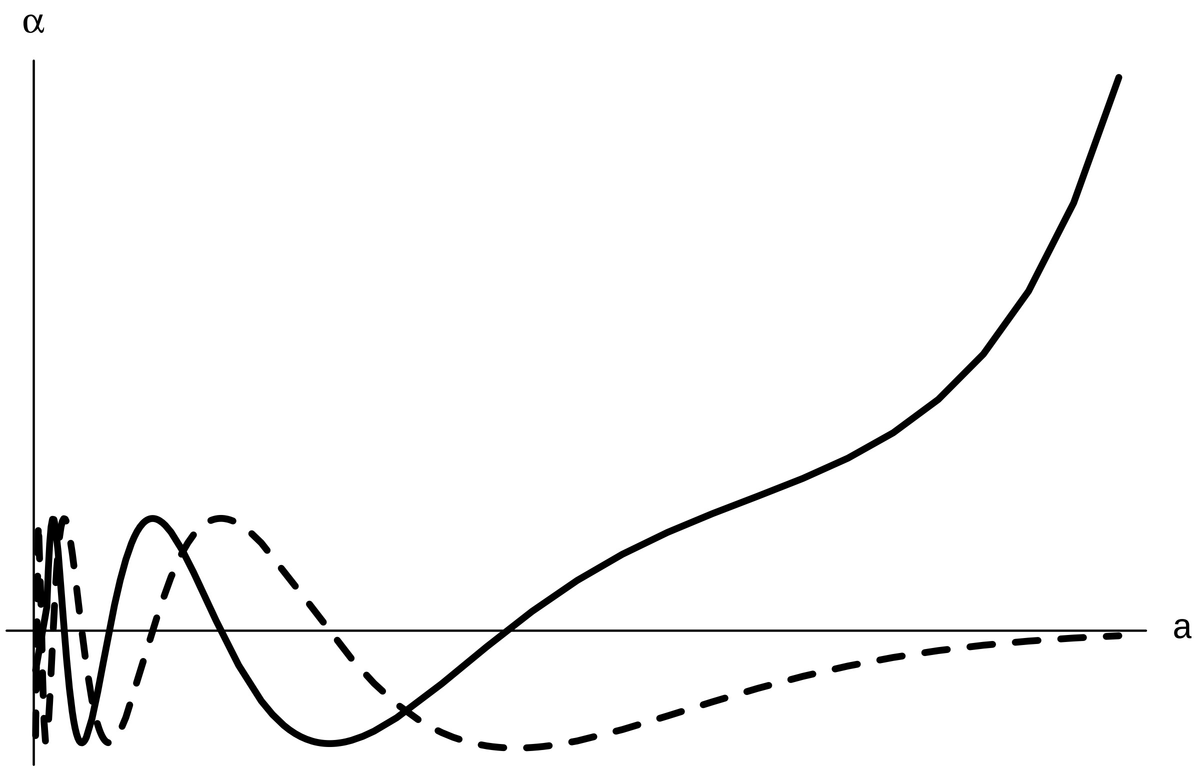

Figure 1: Behaviour of $\alpha(a)$ for $n \in \mathbf{I}$ for the one scalar field case, with $p = 1$, $k = 1$, $A_\alpha = 1$ and $B_\alpha = 0$. The dashed and continous lines represent the imaginary and real parts of α respectively.

k being an integration constant. The solutions for α and β are,

$$\alpha_k(a) = a^{(1-p)/2}\Big[A_\alpha I_n(12a^2) + B_\alpha K_n(12a^2)\Big] \quad , \tag{35}$$

$$\beta_k(\phi) = A_\beta \phi^{(1-m-q)/2} + B_\beta \frac{\phi^{(1+m-q)/2}}{m} \quad , \tag{36}$$

with $n = \frac{\sqrt{(p-1)^2-48k}}{4}$ and $m = \sqrt{(q-1)^2 - 4(3+2\omega)k}$. The function α does not exhibit an oscillatory behaviour unless $n \in \mathbf{I}$. For this case, α oscillates for small values of a, increasing or decreasing for large values of a, suggesting that a classical phase may occur for small values of a only. The function β has an oscillatory behaviour, for all ϕ, if $m \in \mathbf{I}$. In figure 1 we show the behaviour of the real and imaginary parts of α for $n \in \mathbf{I}$. The complete solution of the Wheeler-DeWitt equation is

$$\Psi(a,\phi) = \int A(k)\alpha_k(a)\beta_k(\phi)dk \quad . \tag{37}$$

3.2 Solutions with two scalar fields

For the case where both scalar fields are non null, the Wheeler-DeWitt equation in the minisuperspace reads,

$$\frac{a^2}{12}\Big[\Psi_{aa}+\frac{p}{a}\Psi_a\Big]-\frac{\phi^2}{(3+2\omega)}\Big[\Psi_{\phi\phi}+\frac{q}{\phi}\Psi_\phi\Big]-\Big(\frac{\phi^2}{2}\Big)\Psi_{\xi\xi}=V_\Psi(a)\Psi \quad . \tag{38}$$

We use again the separation of variables method writing $\Psi(a,\phi,\xi)=\Xi(a,\phi)\lambda(\xi)$. Equation (38) separates in two:

$$\begin{aligned}\lambda_{\xi\xi} &= -8k_1\lambda \quad , & (39)\\ \frac{a^2}{12}\Big[\Xi_{aa}+\frac{p}{a}\Xi_a\Big]-\frac{\phi^2}{(3+2\omega)}\Big[\Xi_{\phi\phi}+\frac{q}{\phi}\Xi_\phi\Big] &= V_\Xi(a,\phi)\Xi \quad , & (40)\end{aligned}$$

with $V_\Xi(a,\phi)=48a^4-4k_1\phi^2$, k_1 being an integration constant. Writting $\Xi(a,\phi)=\alpha(a)\beta(\phi)$, we obtain two ordinary equations:

$$\begin{aligned}\alpha_{aa}+\frac{p}{\alpha}\alpha_a &= V_\alpha(a)\alpha \quad , & (41)\\ \beta_{\phi\phi}+\frac{q}{\phi} &= V_\beta(\phi)\beta \quad , & (42)\end{aligned}$$

with

$$\begin{aligned}V_\alpha(a) &= 12\Big(48a^2-\frac{k_2}{a^2}\Big) \quad , & (43)\\ V_\beta(\phi) &= (3+2\omega)\Big(4k_1-\frac{k_2}{\phi^2}\Big) \quad . & (44)\end{aligned}$$

The solutions for α, β and λ are

$$\begin{aligned}\alpha(a) &= a^{(1-p)/2}\Big[A_\alpha I_n(12a^2)+B_\alpha K_n(12\alpha^2)\Big] \quad , & \\ n &= \frac{\sqrt{(p-1)^2-48k_2}}{4} \quad ; & (45)\\ \beta(\phi) &= \phi^{1-q)/2}\Big[A_\beta I_m\Big(2\sqrt{(3+2\omega)k_1}\phi\Big)+B_\beta K_m\Big(2\sqrt{3+2\omega k_1}\phi\Big)\Big] \quad , & \\ m &= \frac{\sqrt{(q-1)^2-4(3+2\omega)k_2}}{2} \quad ; & (46)\\ \lambda(\xi) &= A_\lambda e^{i\sqrt{8k_1}\xi}+B_\lambda e^{-i\sqrt{8k_1}\xi} \quad . & (47)\end{aligned}$$

The coefficients A's and B's are constants. The general solution of the Wheeler-DeWitt equation is

$$\Psi(a,\phi,\xi)=\int A(k_1,k_2)\alpha_{k_2}(a)\beta_{k_1,k_2}(\phi)\lambda_{k_1}(\xi)dk_1dk_2 \quad . \tag{48}$$

In general, α is an exponentially growing or decreasing function of a. If the order of the modified Bessel functions is imaginary, α may exhibit an oscillatory behaviour. However, for these cases, α oscillates for small values of a, increasing or decreasing for large values of a, suggesting again that a classical phase may occur only for small a. This behaviour is displayed in figure 2.

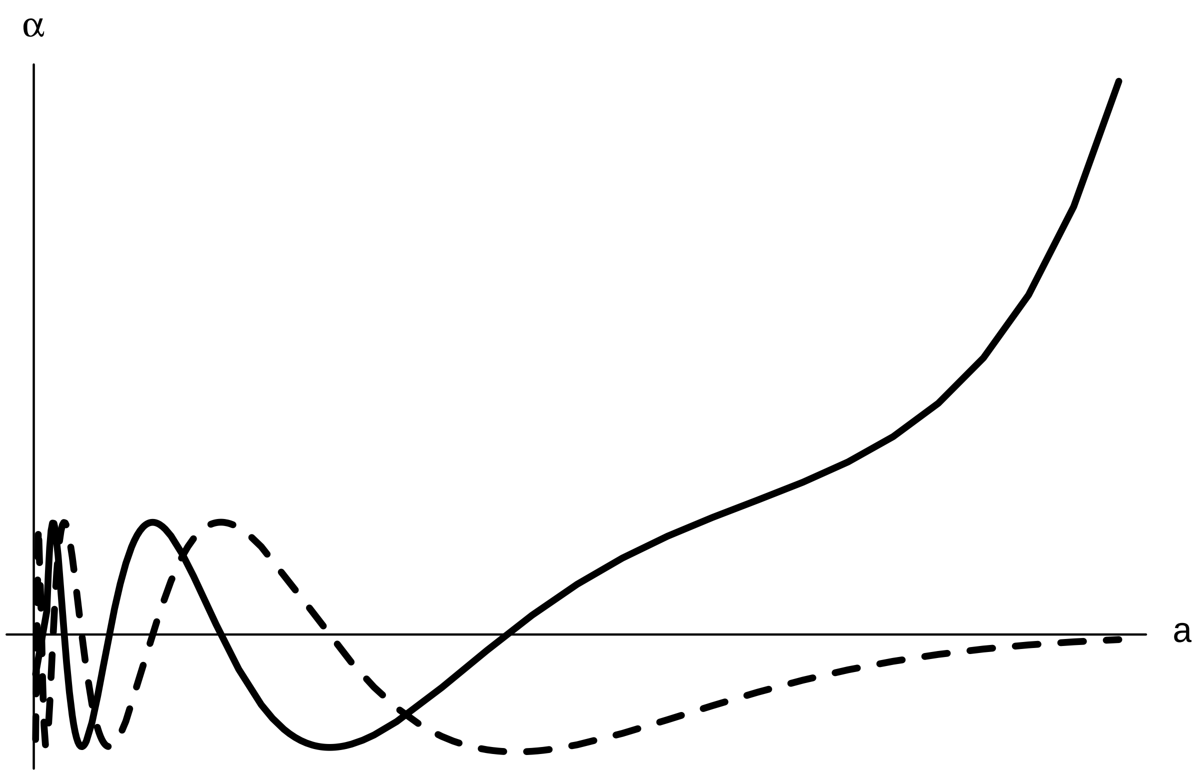

Figure 2: Behaviour of $\alpha(a)$ for the two scalar fields case for $p = 1$, $k_2 = 1$, $A_\alpha = 1$ and $B_\alpha = 0$. The real part is represented by the continous line while the imaginary part is represented by the dashed line.

4 The WKB Approximation

One way to try to obtain the transition to the classical regime from the quantum solutions is to employ the WKB approximation, like in usual quantum mechanics. This is achieved by rewriting the wave function as,

$$\Psi = \exp(\frac{i}{\hbar}S) \quad , \tag{49}$$

substituting it into the Wheeler-DeWitt equation, and performing an expansion in orders of $\hbar$ in S,

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + ... \tag{50}$$

The classical solution must be recovered by constructing a wave packet from S_0:

$$\Psi = \int A(k_0) \exp(\frac{i}{\hbar}S_0) dk_0 \quad , \tag{51}$$

where k_0 is an integration constant. As in the preceding sections, we will analyse the WKB approximation separately for the cases with one and two scalar fields, respectively.

4.1 WKB approximation with one scalar field

In this case, we have $S = S(a, \phi)$, and the WKB expansion in the minisuperspace Wheeler-DeWitt equation, leads to the following equations connecting S_0 and S_1:

$$\frac{a^2}{12}\Big(\frac{\partial S_0}{\partial a}\Big)^2 - \frac{\phi^2}{3+2\omega}\Big(\frac{\partial S_0}{\partial \phi}\Big)^2 + V_\Psi(a) \;=\; 0 \quad ; \tag{52}$$

$$\begin{aligned} \frac{a^2}{12}\Big[i\Big(\frac{\partial^2 S_0}{\partial a^2}\Big) - 2\Big(\frac{\partial S_0}{\partial a}\Big)\Big(\frac{\partial S_1}{\partial a}\Big) + \frac{ip}{a}\Big(\frac{\partial S_0}{\partial a}\Big)\Big] - \\ \frac{\phi^2}{3+2\omega}\Big[i\Big(\frac{\partial^2 S_0}{\partial \phi^2}\Big) - 2\Big(\frac{\partial S_0}{\partial \phi}\Big)\Big(\frac{\partial S_1}{\partial \phi}\Big) + \frac{iq}{\phi}\Big(\frac{\partial S_0}{\partial \phi}\Big)\Big] \;=\; 0 \quad . \end{aligned} \tag{53}$$

First we get a solution for S_0. It can be obtained by taking,

$$S_0(a, \phi) = S_0(a) + S_0(\phi) \quad , \tag{54}$$

leading to two differential equations:

$$\Big(\frac{\mathrm{d}S_0(a)}{\mathrm{d}a}\Big)^2 \;=\; 12\Big(\frac{k_0}{a^2} - 48a^2\Big) \quad , \tag{55}$$

$$\Big(\frac{\mathrm{d}S_0(\phi)}{\mathrm{d}\phi}\Big)^2 \;=\; (3+2\omega)\frac{k_0}{\phi^2} \quad , \tag{56}$$

where k_0 is a separation constant. These equations admit the following analytic solutions:

$$S_0(a) \;=\; \pm\Big[\sqrt{3(k_0 - 48a^4)} - \sqrt{3k_0}\,\mathrm{arctanh}\Big(\sqrt{\frac{k_0 - 48a^4}{k_0}}\Big)\Big] + A_0 \quad , \tag{57}$$

$$S_0(\phi) \;=\; \pm\sqrt{(3+2\omega)k_0}\ln\phi + B_0 \quad , \tag{58}$$

where A_0 and B_0 are integration constants. We follow the same procedure in order to obtain a solution for $S_1(a,\phi)$, considering first $S_1(a,\phi) = S_1(a) + S_1(\phi)$. We get the solutions,

$$S_1(a) = \pm\frac{k_1}{2}\sqrt{\frac{3}{k_0}}\left[\operatorname{arctanh}\left(\sqrt{\frac{k_0-48a^4}{k_0}}\right)+i\frac{p-1}{2}\ln a + \frac{i}{4}\ln(48a^4-k_0)+A_1\right], \quad (59)$$

$$S_1(\phi) = \pm\left[i\frac{q-1}{2} - \frac{k_1}{2}\sqrt{\frac{3+2\omega}{k_0}}\ln\phi\right]+B_1\,, \quad (60)$$

where A_1 and B_1 are integration constants. From the solution for $S_0(a)$, we can easily see that only for $k_0 > 0$ we can obtain an oscillatory behaviour of the wavefunction for small values of a, while for $k_0 < 0$ the wavefunction has an exponential behaviour for any value of a. Similarly, if $(3+2\omega)k_0 > 0$, then $\exp[\frac{i}{\hbar}S_0(\phi)]$ is oscillatory for any value of ϕ, otherwise it has an exponential behaviour. Hence, for $k_0 > 0$ and $\omega > -\frac{3}{2}$, $\exp\frac{i}{\hbar}S_0(a,\phi)$ oscillates for small values of a and any value of ϕ.

We can construct a wavepacket from the above solutions through the expression,

$$\Psi(a,\phi) = \int A(k_0)\exp[\frac{i}{\hbar}S_0(k_0,a,\phi)]dk_0 \quad . \quad (61)$$

where the function $A(k_0)$ is a sharply peaked gaussian centered in $\bar{k}_0$, with width σ. Examining Eq. (57), we can see that $S_0(a)$ becomes very large when a becomes very small. Hence, in the integral (61), constructive interference happens only if

$$\frac{\partial S_0(a,\phi)}{\partial k_0} = 0 \quad , \quad (62)$$

which implies a relation between k_0, a and ϕ, $k_0 = k_0(a,\phi)$. The wave function turns out to be:

$$\Psi(a,\phi) = A[k_0(a,\phi)]\exp\{\frac{i}{\hbar}S_0[k_0(a,\phi),a,\phi]\} \quad . \quad (63)$$

As the gaussian is sharply peaked at $k_0(a,\phi) = \bar{k}_0$, then we obtain that the wave function (63) is sharply peaked at $k_0(a,\phi) = \bar{k}_0$. It can be verified that this relation is exactly the classical relation (13) with $\bar{k}_0$ playing the role of the integration constant A.

4.2 WKB approximation with two coupled scalar fields

We follow the same procedure as before, writing the wave function Ψ in terms of $S(a,\phi,\xi)$, and performing an expansion in orders of $\hbar$. The final equations for S_0 and S_1 are:

$$\frac{a^2}{12}\left(\frac{\partial S_0}{\partial a}\right)^2 - \frac{\phi^2}{3+2\omega}\left(\frac{\partial S_0}{\partial \phi}\right)^2 - \frac{\phi^2}{2}\left(\frac{\partial S_0}{\partial \xi}\right) + V_\Psi(a) = 0 \quad ; \quad (64)$$

$$\begin{aligned}\frac{a^2}{12}\left[i\left(\frac{\partial^2 S_0}{\partial a^2}\right) - 2\left(\frac{\partial S_0}{\partial a}\right)\left(\frac{\partial S_1}{\partial a}\right) + \frac{ip}{a}\left(\frac{\partial S_0}{\partial a}\right)\right] - \\ \frac{\phi^2}{3+2\omega}\left[i\left(\frac{\partial^2 S_0}{\partial \phi^2}\right) - 2\left(\frac{\partial S_0}{\partial \phi}\right)\left(\frac{\partial S_1}{\partial \phi}\right) + \frac{iq}{\phi}\left(\frac{\partial S_0}{\partial \phi}\right)\right] + \\ \frac{\phi^2}{2}\left[i\left(\frac{\partial^2 S_0}{\partial \xi^2}\right) - 2\left(\frac{\partial S_0}{\partial \xi}\right)\left(\frac{\partial S_1}{\partial \xi}\right)\right] = 0 \quad . \end{aligned} \quad (65)$$

Imposing again the ansatz $S_0(a,\phi,\xi) = S_0(a) + S_0(\phi) + S_0(\xi)$, we obtain the following equations:

$$\begin{aligned}
\left(\frac{\partial S_0(a)}{\partial a}\right)^2 &= 12\left(\frac{K_0}{a^2} - 48a^2\right) \quad , && (66)\\
\left(\frac{\partial S_0(\phi)}{\partial \phi}\right)^2 &= (3+2\omega)\left(\frac{K_0}{\phi^2} - k_0\right) \quad , && (67)\\
\left(\frac{\partial S_0(\xi)}{\partial \xi}\right) &= 2k_0 \quad , && (68)
\end{aligned}$$

where K_0 and k_0 are separation constants. The solutions are:

$$\begin{aligned}
S_0(a) &= \pm\Big[\sqrt{3(K_0 - 48a^4)} - \sqrt{3K_0}\mathrm{arctanh}\Big(\sqrt{\frac{K_0 - 48a^4}{K_0}}\Big)\Big] + A_0 \quad ,\\
S_0(\phi) &= \pm\Big[\sqrt{(3+2\omega)(K_0 - k_0\phi^2)} - \sqrt{(3+2\omega)K_0}\mathrm{arctanh}\Big(\sqrt{\frac{K_0 - k_0\phi^2}{K_0}}\Big)\Big] + B_0 \quad ,\\
S_0(\xi) &= \pm\sqrt{2k_0}\xi + C_0,
\end{aligned}$$

where A_0, B_0 and C_0 are integration constants. As in the one scalar field case, we can find solutions for S_1 but they are not important for the construction of the wave packet in our approximation. The solutions S_0 will be enough to recover the classical trajectory. First we note that $K_0 > 0$ leads to a oscillatory behaviour for $\exp\left[\frac{1}{\hbar}S_0(a)\right]$. On the other hand, if $(3+2\omega)K_0 > 0$, keeping $K_0 > 0$, then $\exp\left[\frac{i}{\hbar}S_0(\phi)\right]$ is oscillatory for any value of ϕ when $k_0 < 0$, or only for small values of ϕ when $k_0 > 0$. If $(3+2\omega)K_0 < 0$, then $\exp\left[\frac{i}{\hbar}S_0(\phi)\right]$ has an exponential behaviour for any value of ϕ when $k_0 < 0$ or for small values of ϕ when $k_0 > 0$.

We consider now the superposition given by

$$\Psi(a,\phi,\xi) = \int\int A(k_0,K_0)\exp\frac{i}{\hbar}S_0(a,\phi,\xi,k_0,K_0)dk_0dK_0, \tag{69}$$

where $A(k_0,K_0)$ is a bidimensional gaussian function, centered on $\bar{k}_0 > 0$ and $\bar{K}_0 > 0$ with width σ_1 and σ_2, respectivelly. As before, $S_0(a)$ becomes very large for small a. Hence, we have to guarantee constructive interference by the condition,

$$\left(\frac{\partial S_0(a,\phi,\xi)}{\partial k_0}|_{k_0=\bar{k}_0}\right)^2 + \left(\frac{\partial S_0(a,\phi,\xi)}{\partial K_0}|_{K_0=\bar{K}_0}\right)^2 = 0 \quad . \tag{70}$$

The implicit relations coming from (70) are the same as the classical relations (17,18,19). The classical limit is again recovered only for small a.

5 The Perspective of the Causal Interpretation

In this section, we will apply the rules of the causal interpretation to the wave functions we have obtained in section 3. We first summarize these rules for the case of homogeneous

minisuperspace models. In the case of homogeneous models, the supermomentum constraint $\mathcal{H}^i$ is identically zero, and the shift function N_i can be set to zero without loosing any of the Einstein's equations. The hamiltonian is reduced to general minisuperspace form:

$$H_{GR} = N(t)\mathcal{H}(p^{\alpha}(t), q_{\alpha}(t)), \tag{71}$$

where $p^{\alpha}(t)$ and $q_{\alpha}(t)$ represent the homogeneous degrees of freedom coming from $\Pi^{ij}(x,t)$ and $h_{ij}(x,t)$. The minisuperspace Wheeler-De Witt equation is:

$$\mathcal{H}(\hat{p}^{\alpha}(t), \hat{q}_{\alpha}(t))\Psi(q) = 0. \tag{72}$$

Writing $\Psi = R\exp(iS/\hbar)$, and substituting it into (72), we obtain the following equation:

$$\frac{1}{2}f_{\alpha\beta}(q_{\mu})\frac{\partial S}{\partial q_{\alpha}}\frac{\partial S}{\partial q_{\beta}} + U(q_{\mu}) + Q(q_{\mu}) = 0, \tag{73}$$

where

$$Q(q_{\mu}) = -\frac{1}{R}f_{\alpha\beta}\frac{\partial^2 R}{\partial q_{\alpha}\partial q_{\beta}}, \tag{74}$$

and $f_{\alpha\beta}(q_{\mu})$ and $U(q_{\mu})$ are the minisuperspace particularizations of the DeWitt metric G_{ijkl} [9] and of the scalar curvature density $-h^{1/2}R^{(3)}(h_{ij})$ of the spacelike hypersurfaces, respectively. The causal interpretation, applied to quantum cosmology, states that the trajectories $q_{\alpha}(t)$ are real, independently of any observations. Eq. (73) is the Hamilton-Jacobi equation for them, which is the classical one ammended with a quantum potential term (74), responsible for the quantum effects. This suggests to define:

$$p^{\alpha} = \frac{\partial S}{\partial q_{\alpha}}, \tag{75}$$

where the momenta are related to the velocities in the usual way:

$$p^{\alpha} = f^{\alpha\beta}\frac{1}{N}\frac{\partial q_{\beta}}{\partial t} \ . \tag{76}$$

To obtain the quantum trajectories we have to solve the following system of first order differential equations:

$$\frac{\partial S(q_{\alpha})}{\partial q_{\alpha}} = f^{\alpha\beta}\frac{1}{N}\frac{\partial q_{\beta}}{\partial t} \ . \tag{77}$$

Eqs. (77) are invariant under time reparametrization. Hence, even at the quantum level, different choices of $N(t)$ yield the same spacetime geometry for a given non-classical solution $q_{\alpha}(t)$. There is no problem of time in the causal interpretation of minisuperspace quantum cosmology. Let us then apply this interpretation to our minisuperspace models and choose the gauge $N = 1$.

5.1 One scalar field

The general solution of the Wheeler-DeWitt equation is given by

$$\Psi(a,\phi)=\int A(k)\alpha_k(a)\beta_k(\phi)dk \quad . \tag{78}$$

where

$$\alpha_k = a^{\frac{1-p}{2}}\Big[A_\alpha I_n(12a^2)+B_\alpha K_n(12a^2)\Big] \quad , \tag{79}$$

$$\beta_k = A_\beta \phi^{\frac{1-m-q}{2}}+\frac{B_\beta}{m}\phi^{\frac{1+m-q}{2}} \quad , \tag{80}$$

with

$$n=\frac{\sqrt{(p-1)^2-48k}}{4} \tag{81}$$

and

$$m=\sqrt{(q-1)^2-4(3+2\omega)k} \quad . \tag{82}$$

The momenta are

$$\pi_a = 24a\dot{a} \quad , \tag{83}$$

$$\pi_\phi = -2(3+2\omega)a^3\frac{\dot{\phi}}{\phi^2} \quad . \tag{84}$$

The causal interpretation states that the momenta are also given by

$$\pi_a = \frac{\partial S(a,\phi)}{\partial a} \quad , \tag{85}$$

$$\pi_\phi = \frac{\partial S(a,\phi)}{\partial \phi} \quad , \tag{86}$$

where $S(a,\phi)$ is the total phase of the wave function Ψ. Hence, the Bohmian trajectories will be solutions of the following system of equations:

$$24a\dot{a} = \frac{\partial S(a,\phi)}{\partial a} \quad , \tag{87}$$

$$-2(3+2\omega)a^3\frac{\dot{\phi}}{\phi^2} = \frac{\partial S(a,\phi)}{\partial \phi} \quad . \tag{88}$$

The quantum potential for this problem can be calculated in the usual way. We substitute $\Psi = Re^{iS}$ into the Wheeler-DeWitt equation, obtaining the Hamilton-Jacobi like equation with the extra quantum potential term Q:

$$-\frac{a^2}{12}\Big(\frac{\partial S}{\partial a}\Big)^2+\frac{\phi^2}{3+2\omega}\Big(\frac{\partial S}{\partial \phi}\Big)^2-48a^4+Q=0 \quad , \tag{89}$$

where

$$Q=\frac{1}{R}\Big[\frac{a^2}{12}\Big(\frac{\partial^2 R}{\partial a^2}+\frac{p}{a}\frac{\partial R}{\partial a}\Big)-\frac{\phi^2}{3+2\omega}\Big(\frac{\partial^2 R}{\partial \phi^2}+\frac{q}{\phi}\frac{\partial R}{\partial \phi}\Big)\Big] \quad . \tag{90}$$

Let us apply this interpretation to the simplest case $\Psi = \alpha_k(a)\beta_k(\phi)$. Then the wave-function has the form,

$$\Psi = R_1(a)R_2(\phi)e^{i[S_1(a)+S_2(\phi)]} \quad , \tag{91}$$

since $S(a,\phi) = S_1(a) + S_2(\phi)$. This implies that (87) becomes independent of ϕ. From Eq. (90), we see that $Q(a,\phi) = Q_1(a) + Q_2(\phi)$. To simplify the calculations we set $A_\beta = B_\beta = 0$, and $p = q = 1$. We will first calculate the dynamics of the scale factor when a is small, in order to see if there are singularities. In this approximation we can take just the first term of the series represantion of $I_n(x)$ [10],

$$I_n(x) = \sum_{l=0}^{\infty} \frac{1}{l!\Gamma(n+l+1)}\left(\frac{x}{2}\right)^{n+2l} \quad . \tag{92}$$

For n real, the modified Bessel function $I_n(x)$ is real and the phase of α_k is zero. Hence, the Bohmian equation (87) yields that a is a constant. It is a nonsingular quantum solution but with little physical interest. Hence, in a first moment, we will restrict ourselves to the case where n is a pure imaginary number. Combinations of these two situations will be analyzed afterwards.

In the case where n is pure imaginary, α_k can be written as

$$\alpha_k = c_0 x^{i\nu} = c_0 e^{i\nu \ln x} \quad , \quad n = i\nu = \pm i\sqrt{3k} \quad . \tag{93}$$

The phase and the norm are,

$$\begin{aligned} S_1(a) &= \nu \ln x \quad , \\ R_1(a) &= c_0 \quad . \end{aligned} \tag{94}$$

Defining $x = 12a^2$, Eq. (87) becomes

$$\dot{a} = \frac{\mathrm{d}S}{\mathrm{d}x} = \frac{\nu}{x} \quad , \tag{95}$$

whose solution is

$$a = (\frac{\nu t}{4})^{\frac{1}{3}} \quad . \tag{96}$$

We can see that, in accordance to what was suggested in previous sections, the behaviour of the quantum trajectory is like the classical one for small a, and the singularity will still be present. Note that if ν is positive we have expansion, while if ν is negative we have contraction. For the scalar field, we have:

$$\beta_k = A_\beta \phi^{iu} = B_\alpha e^{iu \ln \phi} \quad , \tag{97}$$

where $u = -\sqrt{(3+2\omega)k}$. The phase and the norm are:

$$\begin{aligned} S_2(\phi) &= u \ln \phi \quad , \\ R_2(\phi) &= B_\alpha \quad . \end{aligned} \tag{98}$$

From Eq. (88) we have,

$$-2\frac{3+2\omega}{\phi^2}a^3\dot{\phi} = \frac{\partial S}{\partial \phi} = \frac{u}{\phi} \quad . \tag{99}$$

Using Eq. (96) we get

$$\phi = t^{\frac{2}{\sqrt{3(3+2\omega)}}} \quad , \tag{100}$$

which is also the classical behaviour for $A = 1$.

It is not surprising that we have obtained the classical behaviour. Since $R_1(a)$ and $R_2(a)$ are constants, the quantum potential is zero, and there is no quantum effect. Note also that solutions (96) and (100) satisfy the hamiltonian constraint with $V(a) = 48a^4$ neglected because a is very small.

For very large a, the Bessel function $I_n(z)$ can be approximated to [10]

$$I_n(x) \sim c_1\frac{e^{12a^2}}{a} \quad , \tag{101}$$

where c_1 is a complex constant. In this case, we have:

$$S_1(a) = \text{const.} \quad , \tag{102}$$

$$R_1(a) = |c_1|\frac{e^{12a^2}}{a} \quad . \tag{103}$$

The quantum trajectory is evidently

$$a = \text{const.} \quad , \tag{104}$$

which is not the classical one. For the scalar field, Eqs. (88,98,104) now yields,

$$\phi = e^{\frac{1}{2}\sqrt{\frac{k}{3+2\omega}}t} \quad , \tag{105}$$

which is the classical behaviour for ϕ in this regime (note that as $a =$ const., $\theta \propto t$ in equation (11)). The different behaviours of the scale factor and the scalar field can be explained with the quantum potential. For the scale factor, Q_1 will be given (see Eq. (90) with $p = 1$):

$$Q_1(a) = 48a^4 + \frac{1}{12} \tag{106}$$

which is of the same size of the classical potential $V(a) = -48a^4$, and hence responsible for this quantum behaviour. For the scalar field, as $R_2(\phi)$ is constant, $Q_2(\phi) = 0$ and the scalar field continues to follow its classical trajectory. Note that the hamiltonian constraint is also satisfied in this limit. Hence we have again obtained the strange result where the classical limit happens only for small values of a.

Let us now take some superpositions of the $\alpha_k(a)$ and $\beta_k(\phi)$ given in Eqs. (79,80). For definitiness, we will choose $p = q = 1$, and $\omega = 0$. We will continue to take only pure imaginary n's. Combinations of real and pure imaginary n's do not change qualitatively the results. The wave function will be given by,

$$\Psi = \sum_{i=1}^{3} I_{n_i}(12a^2)\Big[A_i\phi^{-u_i} + B_i\phi^{u_i}\Big] \tag{107}$$

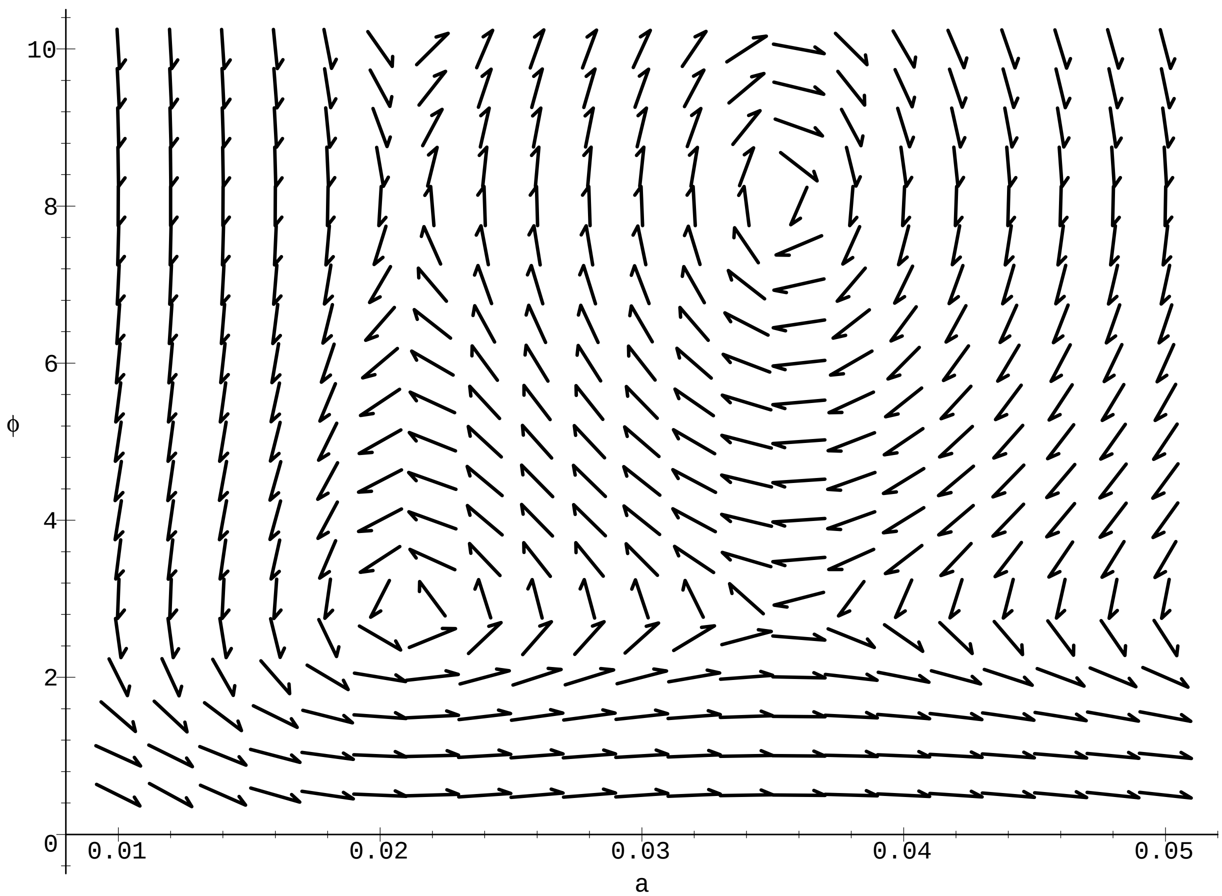

Figure 3: Field plot of a versus ϕ using the causal interpretation for the superpositions of the wave functions in the one scalar field case, in the region of small a.

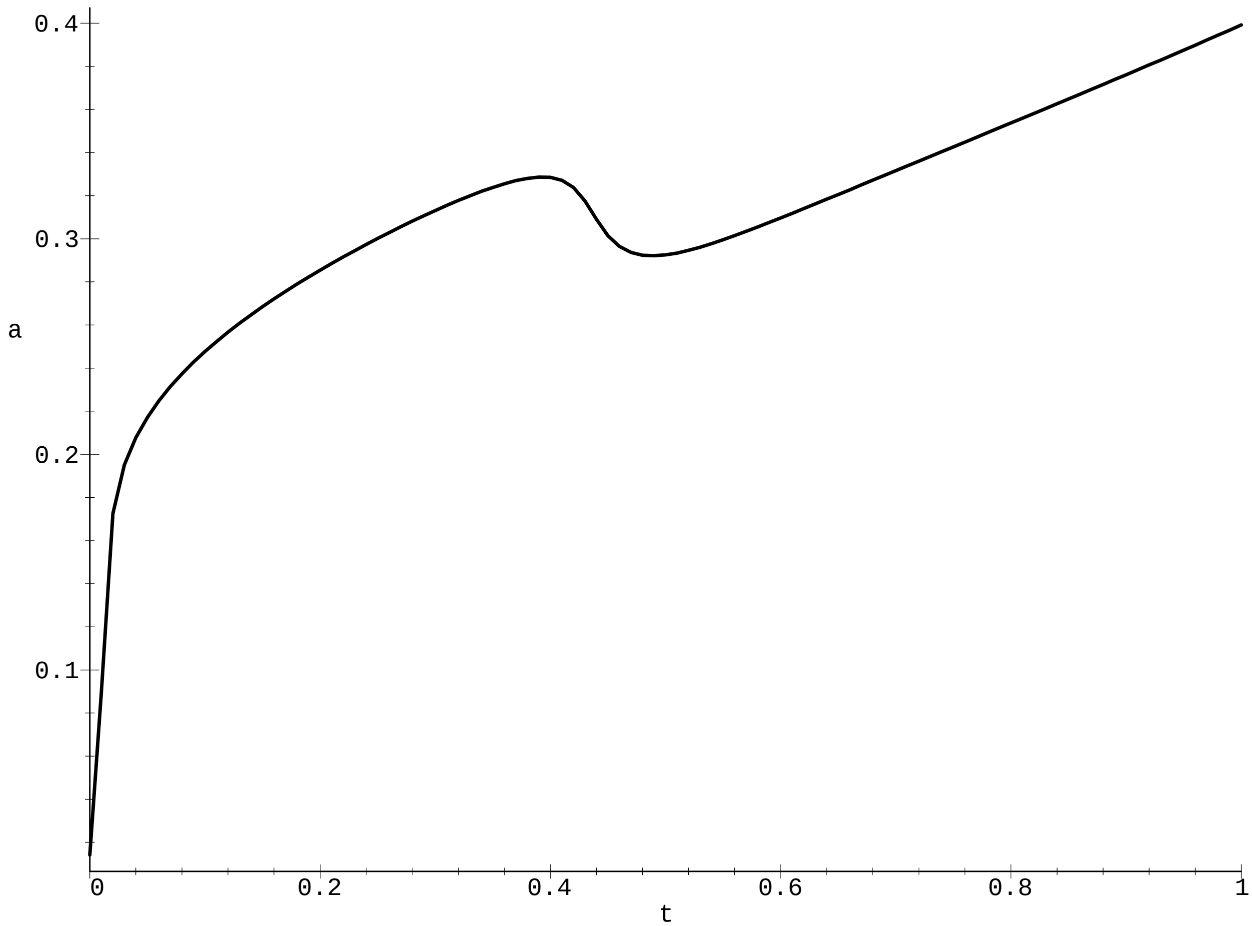

Figure 4: Plot of a particular singular solution for $a(t)$, coming from figure 3, which begins with an inflationary phase.

where

$$\left.\begin{array}{l} k_1 = \frac{1}{3} \rightarrow n_1 = i \quad , \quad u_1 = i \quad , \\ k_2 = k_3 = \frac{e^{2\pi i}}{3} \rightarrow n_2 = n_3 = -i \quad , \quad u_2 = u_3 = -i \quad , \\ A_1 = A_2 = 1 \quad , \quad A_3 = 0 \quad , \\ B_1 = B_2 = 0 \quad , \quad B_3 = 1 \quad . \end{array}\right\} \tag{108}$$

Using Eq. (92) we obtain, for small a:

$$\Psi = \left(\frac{12a^2}{\phi}\right)^i + \frac{1}{(12a^2\phi)^i} + \left(\frac{\phi}{12a^2}\right)^i \quad . \tag{109}$$

Figure 3 shows a field plot of a versus ϕ, for small a, for the Bohmian equations (87,88), with S being the phase of the wave function (109). We can see that there are periodic solutions with very small oscillations around a. They are eternal quantum universes which never grows. The solutions which grow beyond the validity of (109) are singular and inflationary. This can be seen in figure 4.

This result suggests that the initial singularity can be avoided only if we superpose eigenfunctions of opposite frequencies. However, it seems to be be difficult to obtain, in scalar field models, non-singular universes with long expansion period [11].

5.2 Two Scalar Fields

In this case, we have studied the quantum trajectories driven by wave functions obtained from Eq. (48) for some particular $A(k_1, k_2)$.

i) We have fixed $p = q = 1$, $\omega = 0$, $k_1 = -\frac{1}{12}$, $k \equiv k_2$, $A_\alpha = A_\beta = 0$, $B_\alpha = B_\beta = A_\lambda = B_\lambda = 1$ and $A(k_1, k_2) = \frac{3}{2}\delta(k_1 + \frac{1}{12})\tanh(\pi\sqrt{3k_2})$. Using a result of Ref. [10], we obtain,

$$\begin{aligned} \Psi(x, \phi, \xi) &= \cosh\left(\sqrt{\frac{2}{3}}\xi\right)\int_0^\infty \frac{3}{2}\tanh(\pi\sqrt{3k})K_{i\sqrt{3k}}(x)K_{i\sqrt{3k}}(i\phi)dk \quad , \\ &= \cosh\left(\sqrt{\frac{2}{3}}\xi\right)\frac{\pi}{2}\sqrt{\frac{x\phi}{x^2+\phi^2}}e^{-x}\exp\left\{i\left[\frac{\pi}{4} - \phi - \arctan(\frac{\phi}{x})\right]\right\} \quad . \end{aligned} \tag{110}$$

The quantum trajectories can be calculated from the following equations (in the gauge $N = 1$):

$$\begin{array}{l} \pi_a = 24a\dot{a} = \frac{\partial S}{\partial a} = \frac{24a\phi}{x^2+\phi^2} \quad , \\ \pi_\phi = -6\frac{a^3\dot{\phi}}{\pi^2} = \frac{\partial S}{\partial \phi} = \frac{x^2+\phi^2+x}{x^2+\phi^2} \quad , \\ \pi_\xi = -4\frac{a^3\dot{\xi}}{\phi^2} = \frac{\partial S}{\partial \xi} = 0. \end{array} \tag{111}$$

The solutions are:

$$\begin{array}{l} a = \frac{1}{\sqrt{12}}\left[\ln\left(\frac{C}{\sqrt{1+4\eta^2}}\right)\right]^{\frac{1}{2}} \quad , \\ \phi = -\frac{1}{2\eta}\ln\left(\frac{C}{\sqrt{1+4\eta^2}}\right) = -6\frac{a^2}{\eta} \quad , \\ \xi = \text{const.} \quad , \end{array} \tag{112}$$

where $\eta = \int \frac{dt}{a}$ is the conformal time and C is an integration constant. For small a, when η approaches $\pm\frac{\sqrt{c^2-1}}{2}$, these functions tends to:

$$\begin{aligned} & a(t) \propto t^{\frac{1}{3}} \quad , \\ & \phi(t) \propto t^{\frac{2}{3}} \propto a^2 \quad , \\ & \xi = \text{const.} \quad . \end{aligned} \tag{113}$$

which is exactly the classical behaviour for $\omega = 0$. When a is not small, the trajectories are not classical (compare (112) with Eqs. (14,15,16)). This can be seen by inspecting the quantum potential. For two scalar fields it is given by

$$Q = \frac{1}{R}\Big[\frac{a^2}{12}\Big(\frac{\partial^2 R}{\partial a^2} + \frac{p}{a}\frac{\partial R}{\partial a}\Big) - \frac{\phi^2}{3+2\omega}\Big(\frac{\partial^2 R}{\partial \phi^2} + \frac{q}{\phi}\frac{\partial R}{\partial \phi}\Big) - \frac{\phi^2}{2}\frac{\partial^2 R}{\partial \xi^2}\Big] \quad . \tag{114}$$

For our particular problem, we obtain

$$Q = \frac{1}{3}\frac{(x^4 - 2\phi^2 x + \phi^4)}{x^2 + \phi^2} \quad . \tag{115}$$

For small a we have $Q \propto a^2$ (remember that $\phi \propto a^2$ in this limit). In this domain, the kinetic terms dominate:

$$K_a = \frac{a^2\pi_a^2}{12} = \frac{a^2}{12}\Big(\frac{\partial S}{\partial a}\Big)^2 = \frac{1}{3}\frac{x^2\phi^2}{(x^2+\phi^2)^2} \propto \text{const.} \quad , \tag{116}$$

$$K_\phi = -\frac{\phi^2\pi_\phi^2}{3} = -\frac{\phi^2}{3}\Big(\frac{\partial S}{\partial \phi}\Big)^2 = -\frac{\phi^2}{3}\Big(\frac{x^2+\phi^2+x}{x^2+\phi^2}\Big)^2 \propto \text{const.} \quad . \tag{117}$$

Hence the quantum potential becomes negligible when compared with the classical kinetic terms. For a not small, for instance, when $n \to 0$ yielding $a \to a_{max}$ and $\phi \to \infty$, the quantum potential diverges,

$$Q \propto \phi^2 \quad , \tag{118}$$

while the classical potential and kinetic terms behave like

$$K_a \propto \frac{1}{\phi^2} \quad , \tag{119}$$

$$K_\phi \propto \phi^2 \quad , \tag{120}$$

$$V_{cl} \propto a^4 \quad . \tag{121}$$

Hence, together with K_ϕ, the quantum potential becomes the more important term. This behaviour of the quantum potential explains why the trajectories are classical for small a and quantum otherwise.

ii) Let us now take $p = q = 1$, $\omega = 0$, $k_1 = \frac{i}{6}$, $k \equiv k_2$, $A_\alpha = A_\beta = B_\lambda = 0$, $B_\alpha = B_\beta = A_\lambda = 1$ and $A(k_1,k_2) = \delta(k_1 - \frac{i}{6})\sinh(\pi\sqrt{3k_2})K_{i\sqrt{3k_2}}(\sqrt{2}e^{i\frac{\pi}{4}})$. Then we obtain the following wave function (see Ref. [10]):

$$\Psi = \frac{\pi^2}{4}\exp\Big[-\frac{x}{2}(\phi + \frac{1}{\phi}) - \sqrt{\frac{2}{3}}\xi\Big]\exp\Big[i\Big(\sqrt{\frac{2}{3}}\xi - \frac{\phi}{x}\Big)\Big] \quad . \tag{122}$$

The equations of motion are:

$$
\begin{aligned}
\pi_a &= 24a\dot{a} = \frac{\partial S}{\partial a} = 24\frac{a\phi}{x^2} \quad , \\
\pi_\phi &= -6\frac{a^3\dot{\phi}}{\phi^2} = \frac{\partial S}{\partial \phi} = -\frac{1}{x} \quad , \\
\pi_\xi &= -4\frac{a^3\dot{\xi}}{\phi^2} = \frac{\partial S}{\partial \xi} = \sqrt{\frac{2}{3}} \quad .
\end{aligned}
\tag{123}
$$

The solutions are,

$$
\begin{aligned}
a &= t^{\frac{1}{3}} \quad , \\
\phi &= C_0 a^2 = c_0 t^{\frac{2}{3}} \quad , \\
\xi &= -|c_1| t^{\frac{4}{3}} + C_2 \quad .
\end{aligned}
\tag{124}
$$

Note again that these solutons approach the classical one when a is small but are completely different when a is large. Once again, this can be explained by the behaviour of the quantum potential when compared with the kinetic and classical potential terms. The quantum potential is given by

$$
Q = 48a^4 - \frac{\phi^2}{3} \quad . \tag{125}
$$

The kinetic and classical potential terms are given by,

$$
\begin{aligned}
K_a &= \frac{\phi^2}{3x^2} \quad , \\
K_\phi &= -\frac{\phi^2}{3x^2} \quad , \\
K_\xi &= \frac{\phi^2}{3} \quad , \\
V_a &= -48a^4 \quad .
\end{aligned}
\tag{126}
$$

For small a, Q, V_a and K_ξ goes to zero while K_a and K_ϕ are constant. For large a, Q, V_ϕ and K_ξ become comparable and large, while K_a and K_ϕ continue to be constant. Hence, in this situation, the quantum potential becomes important, driving the quantum behaviour of the Bohmian trajectories. Note that the sum of (125) with (126) gives zero because the Bohmian trajectories must satisfy the hamiltonian constraint ammended with the quantum potential term. We have also calculated the Bohmian trajectories for other exact wave solutions of the Wheeler-DeWitt equation. All of them present the same behaviour.

We conclude this section by stating that in quantum cosmology it is not necessary that the classical behaviour appears when a is large, while quantum behaviour is present when a is small. It can indeed be the reverse. This was already pointed out in [12] and we presented specific examples illustrating this fact. It should also be commented that the result of this seciton using the causal interpretation are in qualitative agreement with the results of the previous section.

6 Conclusion

In this paper, we have studied classical and quantum minisuperspace models containing one and two scalar fields. We have shown that all classical solutions are singular. After quantizing the models, we have obtained the general solution of the Wheeler-De Witt

equation. Usually, the solutions are oscillatory when the scale factor is small and not oscillatory when the scale factor becomes larger. This suggests that non-classical behaviour may occur when the sacale factor is large. We studied gaussian superpositions of WKB wave functions to investigate if they correspond to quasi-classical states, as suggested in Ref. [13]. We have shown that indeed these wave functions are peaked around the classical trajectories in configuration space, but only for small a.

After, we applied the causal interpretation of quantum mechanics to these models. In this interpretation, it is possible to calculate quantum trajectories, independently of any observations. We have shown that the trajectories calculated following this interpretation usually present the classical behaviour when the scale factor is small and non-classical behaviour when a is large, as suspected. This means that these quantum trajectories still presents an initial singularity. We have also seen that if we superpose negative with positive frequency solutions, then we can find trajectories which are no more classical for small a. We can have eternal periodic quantum universes with very small oscilations. These universes, however, never scape the Planck length. There are also singular solutions with a short period of inflation which grow forever. We could not find any non-singular solution which grows to the size of our universe, with a classical limit for large a.

The fact that quantum behaviour happens when a is large is not surprising. It was already obtained in Ref. [11] and suggested to exist when the scale factor grows like $t^{1/3}$ in Ref. [12], which is our case. Hence, in quantum cosmology it is not necessarily true that large scale factors implies classical behaviour. For the scalar field models we have analyzed in this paper, the reverse seems to be more usual. It means that it is possible to have in our universe some degrees of freedom which still behave quantum mechanically in spite of it being very big. This gives us some hope of being possible to detect or experience quantum cosmological effects in the real universe we live in, bringing quantum cosmology to the realm of testable physical theories. The problem should be to find which degrees of freedom can possess this property. To do this, we should improve this minisuperspace model with the accretion of small inhomogeneous perturbations, which contain an infinity number of degrees of freedom, and see what happens with the new inhomogeneous degrees of freedom. We will get closer to the real universe but we will have to face new technical and interpretational problems. This will be the subject of our future investigations.
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